Economic Growth, June 2015.

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A suggested solution to the problem set at the exam in Economic Growth, June 4, 2015

(3-hours closed book exam)[1]

As formulated in the course description, a score of 12 is given if the student's performance demonstrates precise understanding of the concepts and methods needed for analyzing the factors that matter for economic growth.

1. Solution to Problem 1 (35 %)

We consider an economy with aggregate production function

$$Y_t = \tilde{F}(K_t, L_t, t),$$

where $\tilde{F}$ is a neoclassical production function w.r.t. K and L, Y is GNP, K is capital input, and L is labor input. Time is continuous. For any variable z which is a differentiable function of time, t, we apply the notation $g_z \equiv \dot{z}/z$, where $\dot{z} \equiv dz/dt$.

a) We take the total derivative of $\tilde{F}$ w.r.t. t:

$$\dot{Y}_t = \tilde{F}_K(K_t, L_t, t)\dot{K}_t + \tilde{F}_L(K_t, L_t, t)\dot{L}_t + \tilde{F}_t(K_t, L_t, t) \cdot 1.$$

Dividing through by Y_t gives

$$\begin{aligned} g_{Yt} &\equiv \frac{\dot{Y}_t}{Y_t} = \frac{1}{Y_t}\left[\tilde{F}_K(K_t, L_t, t)\dot{K}_t + \tilde{F}_L(K_t, L_t, t)\dot{L}_t + \tilde{F}_t(K_t, L_t, t) \cdot 1\right] \\ &= \frac{K_t\tilde{F}_K(K_t, L_t, t)}{Y_t} g_{Kt} + \frac{L_t\tilde{F}_L(K_t, L_t, t)}{Y_t} g_{Lt} + \frac{\tilde{F}_t(K_t, L_t, t)}{Y_t} \\ &\equiv \varepsilon_{Kt} g_{Kt} + \varepsilon_{Lt} g_{Lt} + \frac{\tilde{F}_t(K_t, L_t, t)}{Y_t}, \end{aligned}$$

where ε_{Kt} and ε_{Lt} are shorthands for $\varepsilon_K(K_t, L_t, t) \equiv \frac{K_t\tilde{F}_K(K_t,L_t,t)}{\tilde{F}(K_t,L_t,t)}$ and $\varepsilon_L(K_t, L_t, t) \equiv \frac{L_t\tilde{F}_L(K_t,L_t,t)}{\tilde{F}(K_t,L_t,t)}$, respectively, that is, the partial output elasticities w.r.t. the two production

[1] The solution below contains *more* details and more precision than can be expected at a three hours exam. The percentage weights should only be regarded as indicative. The final grade will ultimately be based on an assessment of the quality of the answers to the exam questions in their totality.

factors, respectively, evaluated at the factor combination (K_t, L_t) at time t. The TFP *growth rate* is the residual

$$x_t \equiv g_{\text{TFP}t} \equiv g_{Yt} - (\varepsilon_{Kt} g_{Kt} + \varepsilon_{Lt} g_{Lt}) = \frac{F_t(K_t, L_t, t)}{Y_t}, \tag{1.1}$$

The TFP growth rate is thus that fraction of current output growth that is *not attributable* to current growth in the capital and labor inputs. Under the assumption of full capacity utilization, the TFP growth rate can thus be interpreted as reflecting the contribution to current output growth from current technical change (in a broad sense including learning by doing and organizational improvement).

From now on we assume that $\tilde{F}$ has CRS w.r.t. K and L. We let $y_t \equiv Y_t/L_t$ and $k_t \equiv K_t/L_t$.

b) In view of CRS, from Euler's theorem follows $\varepsilon_{Kt} + \varepsilon_{Lt} = 1$. So (1.1) gives

$$\begin{aligned} x_t &\equiv g_{Yt} - (\varepsilon_{Kt} g_{Kt} + (1 - \varepsilon_{Kt}) g_{Lt}) = g_{Yt} - g_{Lt} - \varepsilon_{Kt}(g_{Kt} - g_{Lt}) \\ &= g_{yt} - \varepsilon_{Kt} g_{kt}. \end{aligned}$$

Hence,

$$g_{yt} = x_t + \varepsilon_{Kt} g_{kt}. \tag{1.2}$$

We now also assume that $\tilde{F}$ can be written

$$\tilde{F}(K_t, L_t, t) = F(K_t, A_t L_t), \tag{*}$$

where the technology level A_t grows at a given constant rate $g > 0$ and employment grows at a given constant rate $n > 0$. Moreover, the increase in capital per time unit is given by

$$\dot{K}_t = S_t - \delta K_t \equiv Y_t - C_t - \delta K_t, \qquad \delta \geq 0, \tag{**}$$

where C is aggregate consumption and not all of Y is consumed.

c) From now on I skip the explicit dating of the variables. By $Y = F(K, AL)$ and CRS follows that

$$1 = F(\frac{K}{Y}, \frac{AL}{Y}). \tag{1.3}$$

And from the given information, we know from the balanced growth equivalence theorem that along a BGP, Y/K is constant, hence $g_Y = g_K$. By (1.3) and constancy of K/Y along a BGP follows that AL/Y is constant along a BGP, whereby

$$g_Y = g_K = g + n.$$

d) So, along the BGP we also have

$$g_y = g_Y - n = g. \tag{1.4}$$

Yes, there is a sense in which technical progress, along the BGP, explains more than what the growth accounting under a) and b) suggests. Indeed, while (1.4) shows that the whole of g_y is due to technical progress, the growth accounting under a) and b) just gave (1.2), suggesting that technical progress only explains a part of g_y, the remainder being due to growth in the capital-labor ratio k_t. The point is that along the BGP also the growth in k_t is due to technical progress since $g_k = g_K - n = g_Y - n = g$.

e) In view of CRS, (*) gives

$$Y = AL \cdot F(\frac{K}{AL}, 1) \equiv AL \cdot f(\tilde{k}),$$

where f is the production function in intensive form and $\tilde{k} \equiv K/(AL)$. Under perfect competition, the *labor income share* is

$$SL \equiv \frac{wL}{Y} = \frac{(f(\tilde{k}) - kf'(\tilde{k}))AL}{f(\tilde{k})AL} = \frac{f(\tilde{k}) - kf'(\tilde{k})}{f(\tilde{k})} \equiv \frac{\tilde{w}(\tilde{k})}{f(\tilde{k})} \equiv SL(\tilde{k}),$$

as was to be shown.

Yes, $SL = SL(\tilde{k})$ will necessarily be constant along the BGP because $\tilde{k}$ will be constant along the BGP. Indeed,

$$\frac{Y}{K} = \frac{f(\tilde{k})AL}{\tilde{k}AL} = \frac{f(\tilde{k})}{\tilde{k}},$$

and since Y/K is constant along the BGP, so must $\tilde{k}$ be in view of $f'' < 0$.

f) We have

$$SL \equiv \frac{wL}{Y} = \frac{wL}{\hat{r}K + wL} = \frac{w/A}{\hat{r}K/(AL) + w/A} \equiv \frac{\tilde{w}}{\hat{r}\tilde{k} + \tilde{w}} = \frac{\frac{\tilde{w}/\hat{r}}{\tilde{k}}}{1 + \frac{\tilde{w}/\hat{r}}{\tilde{k}}}, \tag{1.5}$$

where the second equality follows from perfect competition and Euler's theorem. Moreover, under perfect competition, firms are cost-minimizing, hence, from the given information, we have

$$\sigma(\tilde{k}) = \mathrm{E}\ell_{\tilde{w}/\hat{r}}\tilde{k}. \tag{1.6}$$

According to Piketty's interpretation of the empirical evidence, $\sigma(\tilde{k}) > 1$. At the same time Piketty predicts that $\tilde{k}$ will be rising and SL falling. (This is essentially all the syllabus says about Piketty; hence, the questions f) and g) are not intended to be a test of the understanding of Piketty but a test of the understanding of neoclassical thinking.)

In neoclassical thinking (which need not be the same as Piketty's), if $\sigma(\tilde{k}) > 1$, then, by (1.6), we get the forecast that the relative factor price $\tilde{w}/\hat{r}$ will rise less fast than $\tilde{k}$, and so $\frac{\tilde{w}/\hat{r}}{\tilde{k}}$ will be falling. In (1.5) the numerator will thus be falling faster than the denominator, and so SL *will* be falling.

The odd aspect is, however, that the *dominant* interpretation of the empirical evidence is that $\sigma(\tilde{k}) < 1$. In this case, the neoclassical forecast, based on (1.5) and (1.6), is that a rising $\tilde{k}$ leads to a *rising* labor income share. Indeed, when $\mathrm{E}\ell_{\tilde{w}/\hat{r}}\tilde{k} < 1$, the relative factor price $\tilde{w}/\hat{r}$ rises faster than $\tilde{k}$ and so the numerator in (1.5) will rise faster than the denominator.

2. Solution to Problem 2 (55 %)

Aggregate output is Y_t per time unit and output is used for private consumption, $C_t \equiv c_t L$, the public service, G_t, and investment, I_t, in (physical) capital, i.e., $Y_t = C_t + G_t + I_t$. The aggregate stock of capital, K_t, changes according to $\dot{K}_t = I_t - \delta K_t$, where $\delta \geq 0$. The initial value $K_0 > 0$ is given. The capital stock in the economy is owned, directly or indirectly (through bonds and shares), by the households. Markets are competitive. The equilibrium real wage is denoted w_t. There is a perfect market for loans with a real interest rate, r_t, and there is no uncertainty.

There is a given tax rate, τ, on private financial wealth. Aggregate private financial wealth is denoted V_t and equals the aggregate capital stock since there is no government debt and natural resources are ignored. The service G_t is the only public expenditure and the government budget is balanced at every t:

$$G_t = \tau V_t = \tau K_t, \qquad \tau > 0. \tag{GBC}$$

The production function for firm i is

$$Y_{it} = A K_{it}^{\alpha} (G_t L_{it})^{1-\alpha}, \quad A > 0, 0 < \alpha < 1, \qquad i = 1, 2, ..., M. \tag{*}$$

a) (*), together with (GBC), indicates that G is a government service that affects productivity. Since the productivity of every worker depends on the total of G (not the per capita amount, G/L), G is completely nonrival. We might think of government-provided software and website information about how to use new technologies. We might think of G as technology information service and teaching transmitted via the internet. From (GBC) we see there is no fee for using G.

b) The results to be explained are that in equilibrium

$$r_t = \alpha A(\tau L)^{1-\alpha} - \delta \equiv r, \tag{2.1}$$

and

$$Y_t = A(\tau L)^{1-\alpha} K_t \equiv \bar{A}(\tau) K_t, \tag{2.2}$$

respectively.

From now on I skip the explicit dating of the variables unless needed for clarity. At any date firm i maximizes current pure profits, $\Pi_i = AK_i^\alpha (GL_i)^{1-\alpha} - \hat{r}K_i - wL_i$, where $\hat{r}$ is the capital cost per unit of capital service. A first-order condition for an interior solution is

$$\partial \Pi_i / \partial K_i = \alpha A K_i^{\alpha-1} (GL_i)^{1-\alpha} - \hat{r} = \alpha A k_i^{\alpha-1} G^{1-\alpha} - \hat{r} = 0, \qquad i = 1, 2, \ldots, M, \tag{2.3}$$

where $k_i \equiv K_i / L_i$. From this follows that the profit-maximizing capital-labor ratio, k_i, will be the same for all firms, say $\bar{k}$. In factor market equilibrium, at any t we must then have $\bar{k}_t = K_t / L \equiv k_t$, where k_t is pre-determined and given from the supply side. Solving for $\hat{r}_t$, we thus get in equilibrium

$$\begin{aligned} \hat{r}_t &= r_t + \delta = \alpha A k_t^{\alpha-1} G_t^{1-\alpha} = \alpha A K_t^{\alpha-1} (G_t L)^{1-\alpha} = \alpha A K_t^{\alpha-1} (\tau K_t L)^{1-\alpha} \\ &= \alpha A (\tau L)^{1-\alpha} \qquad \text{or} \qquad r_t = \alpha A (\tau L)^{1-\alpha} - \delta \equiv r, \end{aligned}$$

in accordance with (2.1).

As to (2.2) we have

$$\begin{aligned} Y_t &= \sum_i Y_{it} = \sum_i y_{it} L_{it} = A k_t^\alpha G_t^{1-\alpha} \sum_i L_{it} = A k_t^\alpha G_t^{1-\alpha} L = A K_t^\alpha (G_t L)^{1-\alpha} \\ &= A K_t^\alpha (\tau K_t L)^{1-\alpha} = A (\tau L)^{1-\alpha} K_t \equiv \bar{A}(\tau) K_t, \end{aligned} \tag{2.4}$$

as was to be shown.

c) We must distinguish between the before-tax and after-tax rate of return on a household's financial wealth. The before-tax rate of return is r and the after-tax rate of return is $r - \tau$.

d) Given $\rho > 0$ and $\theta > 0$, the individual household solves

$$\begin{aligned} \max_{(c_t)_{t=0}^\infty} U_0 &= \int_0^\infty \frac{c_t^{1-\theta}}{1-\theta} e^{-\rho t} dt \qquad \text{s.t.} \\ c_t &\geq 0, \\ \dot{a}_t &= (r - \tau) a_t + w_t - c_t, \qquad a_0 \quad \text{given}, \qquad (2.5) \\ \lim_{t \to \infty} a_t e^{-(r-\tau)t} &\geq 0. \qquad \text{(NPG)} \end{aligned}$$

The current-value Hamiltonian is

$$H = \frac{c^{1-\theta}}{1-\theta} + \lambda\left[(r-\tau)a + w - c\right],$$

where λ can be interpreted as the shadow price of per capita financial wealth along the optimal path. First-order conditions are

$$\partial H/\partial c = c^{-\theta} - \lambda = 0, \text{ i.e., } c^{-\theta} = \lambda, \tag{2.6}$$

$$\partial H/\partial K = \lambda(r-\tau) = \rho\lambda - \dot{\lambda}, \text{ i.e., } r - \tau - \rho = -\dot{\lambda}/\lambda, \tag{2.7}$$

and the necessary transversality condition is

$$\lim_{t\to\infty} a_t e^{-(r-\tau)t} = 0. \tag{TVC}$$

Log-differentiation w.r.t. t in (2.6) and inserting into (2.7) gives the Keynes-Ramsey rule for this model:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r-\tau-\rho) = \frac{1}{\theta}\left[\alpha A(\tau L)^{1-\alpha} - \delta - \tau - \rho\right] \equiv g_c. \tag{2.8}$$

Comment: We see that the growth rate of per capita consumption is a constant "from the beginning".

e) The constancy "from the beginning" of g_c is a first suggestion that the model may be a reduced-form AK model. As (2.4) shows that along an equilibrium path, aggregate output is proportional to the aggregate stock of capital, this suggestion is confirmed. From the general theory of reduced-form AK models with Ramsey households we know that per capita capital, $k_t \equiv K_t/L$, will grow at the same constant rate as per capita consumption already "from the beginning" (no transitional dynamics). And by (2.4), so will then $y_t \equiv Y_t/L = \bar{A}(\tau)k_t$. Hence,

$$g_k = g_y = g_c = \frac{1}{\theta}\left[\alpha A(\tau L)^{1-\alpha} - \delta - \tau - \rho\right]. \tag{2.9}$$

To ensure boundedness of the utility integral U_0, we need

$$\rho > (1-\theta)g_c, \tag{A1}$$

where g_c is given by (2.9). To ensure positive growth, we need

$$\rho + \delta < \alpha A(\tau L)^{1-\alpha} - \tau. \tag{A2}$$

As long as (A2) is satisfied, so is (A1) if for instance $\theta \geq 1$, since $\rho > 0$ according to the given information. Fig. 2.1 illustrates that it is always possible to choose τ small enough

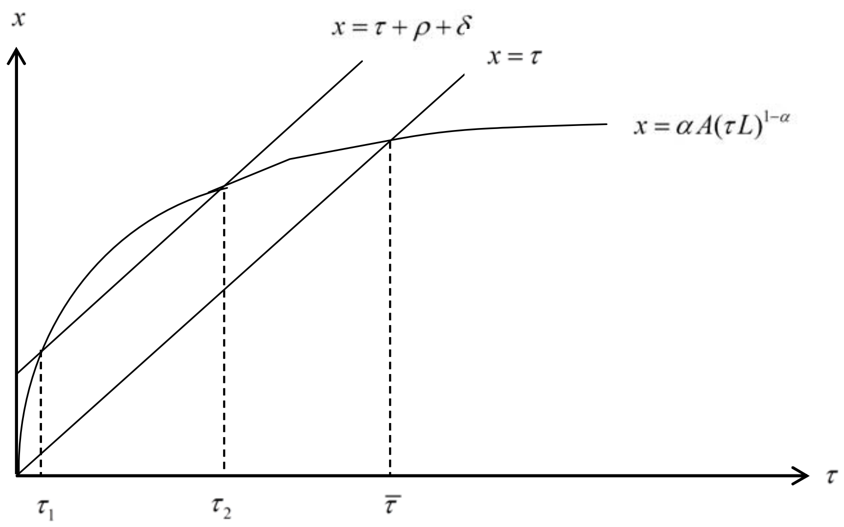

Figure 2.1:

so that $\alpha A(\tau L)^{1-\alpha} - \tau > 0$. This provides scope for $0 < \rho + \delta < \alpha A(\tau L)^{1-\alpha} - \tau$ to be satisfied when $0 < \tau_1 < \tau < \tau_2 < \bar{\tau}$, cf. Fig. 2.1.

f) By (2.9),

$$\begin{aligned} \frac{\partial g_y}{\partial L} &= \frac{1}{\theta}(1-\alpha)\alpha A \tau^{1-\alpha} L^{-\alpha} > 0, \\ \frac{\partial g_y}{\partial \tau} &= \frac{1}{\theta}\left[(1-\alpha)\alpha A L^{1-\alpha}\tau^{-\alpha} - 1\right] \gtreqqless 0, \text{ that is,} \\ \text{for } \tau &\lesseqqgtr \left[(1-\alpha)\alpha A L^{1-\alpha}\right]^{1/\alpha} \equiv \tau^*. \end{aligned}$$

Intuition:

Larger L leads to larger growth rate because of the economies of scale implied by the non-rivalry of the produced input G. This strong form of the scale effect is due to the lack of diminishing returns to capital at the aggregate level as exposed by (2.4).

The role of the tax rate τ on financial wealth is more intricate. An increase in τ has two effects that go in opposite directions. On the one hand, a higher τ means a higher level of productive services, G, and therefore a higher Y/K (see second line of (2.4)), which implies a higher growth potential. On the other hand, a higher τ means both more preempting of output and more 'distortion' of saving incentives and therefore *lower* growth. Starting from a low τ, when τ increases, the first effect dominates until $\tau = \tau^*$. A further increase in τ lowers growth because the combined preempting and distortionary effect dominates. This pattern reflects the positive, but diminishing marginal productivity of G (cf. first line of (2.4)).

g) The two mutually related distinctive features of the model are the following. First, the non-rival productive service, G, enters the production function (*) in a very powerful way, namely such that labor efficiency is proportional to G (no diminishing returns to G). This amounts to a questionable knife-edge condition, cf. h) below.

Second, the model assumes that $n = 0$. To deliver reasonable results, the model *has* to assume this because the strong scale effect implied by the mentioned knife-edge condition will in combination with $n > 0$ result in a forever rising per capita growth rate (although not in the extreme form leading to infinite output and consumption in *finite* time; that would require IRS w.r.t. *producible* inputs, while here we have "only" CRS w.r.t. producible inputs, K and G).

These facts motivate the more inclusive alternative specification where $n \geq 0$ and (*) is replaced by the assumption

$$Y_{it} = AK_{it}^{\alpha}(G_t^{\lambda}L_{it})^{1-\alpha}, \qquad A > 0, 0 < \alpha < 1, 0 < \lambda \leq 1, \qquad i = 1, 2, ..., M. \tag{**}$$

h) In the first line of (2.4) we throughout replace G_t by G_t^{λ} and thereby end up with

$$Y_t = AK_t^{\alpha}(G_t^{\lambda}L)^{1-\alpha}.$$

i) We now let $0 < \lambda < 1$. In view of CRS

$$1 = A\left(\frac{K}{Y}\right)^{\alpha}\left(\frac{G^{\lambda}L}{Y}\right)^{1-\alpha}.$$

Along a balanced growth path (BGP) with positive saving, Y/K is constant. Hence, so is $G^{\lambda}L/Y$. Taking growth rates thus gives $\lambda g_G + n = g_Y$. Thereby

$$g_Y = \lambda g_G + n = \lambda g_Y + n, \text{ i.e., } g_Y = \frac{n}{1-\lambda},$$

where the second equality comes from constancy of

$$\frac{G}{Y} = \frac{G/K}{Y/K} = \frac{\tau}{Y/K}, \qquad \text{(by (GBC))}$$

where Y/K is constant along a BGP. We get, along a BGP,

$$g_y = g_Y - n = \frac{n}{1-\lambda} - \lambda = \frac{\lambda n}{1-\lambda}, \tag{2.10}$$

so that

$$\frac{\partial g_y}{\partial \tau} = 0.$$

j) In contrast to the first model version, the tax rate τ no longer affects the long-run growth rate. While the knife-edge condition $\lambda = 1$ (together with $n = 0$) resulted in *fully-endogenous growth*, the more robust case $0 < \lambda < 1$ results in DRS w.r.t. producible inputs (in that $\alpha + \lambda(1 - \alpha) < 1$). This implies that growth is *semi-endogenous* (i.e., per capita growth is generated by an internal mechanism but can only be sustained if supported by growth in an exogenous factor, here population, which allows the economy to "take advantage" of the economies of scale implied by the non-rivalry of the public productive service G).

When $0 < \lambda < 0$, instead of a strong scale effect (scale effect on sustained growth), presence of a weak scale effect (scale effect on the *level* of per capita income and per capita consumption) can be shown.

k) The social planner will at any t want to choose G so as to maximize *net output* defined as

$$Y^n \equiv Y - G = AK^{\alpha}(G^{\lambda}L)^{1-\alpha}.$$

The first-order condition is

$$\frac{\partial Y^n}{\partial G} = \lambda(1-\alpha)\frac{Y}{G} - 1 = 0,$$

implying

$$\frac{G}{Y} = \lambda(1-\alpha) \qquad \text{or} \qquad G = \lambda(1-\alpha)Y.$$

The second-order condition gives $\partial^2 Y^n/\partial G^2 < 0$.

The reason that the social planner will at any t want to choose G so as to maximize *net output,* $Y - G$, is that this means maximization, at any t, of what is available for final use, consumption and investment. This implies static efficiency; the level effect mentioned at j) is maximized. For small G, i.e., $G < \lambda(1-\alpha)Y$, it holds that $\partial Y/\partial G > 1$ so that a higher G raises net output. But due to diminishing marginal productivity of G, G should only be raised up to the level where $\partial Y/\partial G = 1$, i.e., the level $G = \lambda(1-\alpha)Y$. Above this point, $\partial Y/\partial G < 1$ so that there will be a gain by *reducing* G.

3. Solution to Problem 3 (10 %)

a) Formulas like $g_y = \lambda n/(1-\lambda)$ come from semi-endogenous knowledge-driven growth models like Arrow's or Jones'. Here population growth contributes to per capita growth as explained under j) in Problem 2. In view of cross-border diffusion of ideas and technology, this proposition should not be seen as a prediction about individual countries, however. It should rather be seen as pertaining to larger regions, nowadays probably the total industrialized part of the world. So the single country is not the relevant unit of observation and cross-country regression analysis thereby not the right framework for testing such a link from n to g_y.

b) The two expanding input variety models with knowledge spillovers in syllabus share the following aggregate invention "production functions":

$$\dot{N}_t = \eta N_t^{\varphi} L_{Rt}, \qquad \eta > 0, \varphi \leq 1, \tag{3.1}$$

where N_t is the number of existing different varieties of intermediate goods (indivisibilities are ignored), and L_{Rt} is the aggregate input of research labor at time t. The number, N_t, of existing different varieties of intermediate goods can be interpreted as reflecting the stock of technical knowledge. The time derivative, $\dot{N}_t$, reflects the number of new technical designs for intermediate goods, invented per time unit at time t. Thus, $\dot{N}_t$ can be seen as the increase per time unit in technical knowledge. This increase is determined by the input of research labor, L_{Rt}, and the economy-wide research productivity, ηN_t^{φ}, exogenous to the "small" individual R&D firms. For $\varphi \neq 0$, the research productivity depends on the stock of technical knowledge. This dependency is positive if $\varphi > 0$ ("the standing on the shoulders of giants case") and negative if $\varphi < 0$ ("the fishing out case", "the standing on the toes case").

The knife-edge case $\varphi = 1$ (together with $n = 0$, where n is the growth rate of the labor force $L_t = L_{Yt} + L_{Rt}$) gives the Romer version. And the case $\varphi < 1$ (together with $n \geq 0$) gives the Jones version (the calibration by Jones, 1995, suggests $0 < \varphi < 1$).

In the knife-edge case $\varphi = 1$ (together with $n = 0$), fully-endogenous growth is generated, and there i strong scale effect on growth. In the robust case $\varphi < 1$ there is sustained semi-endogenous growth if $n > 0$. Instead of the empirically problematic strong scale effect there is a weak scale effect, which has some empirical support.

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